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THE LINEAR STABILITY OF GENERAL TWO-PHASE FLOW MODELS—II

A. PROSPERETTI†
Institute of Physics, University of Milan, Milan, Italy

A. V. JONES
CEC Joint Research Centre, Ispra Establishment, 21020 Ispra (Va), Italy

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Abstract—The technique for the linear stability analysis of two-phase flow models developed in a previous paper has been extended to deal with models containing space and time derivatives of the flow variables of arbitrary order. A number of current models for the description of stratified flow have been analysed in detail as an illustration of the utility of the technique. The role of interphase drag functions for long wavelengths and of surface tension for short wavelengths clearly emerges from these examples. Effects such as transverse momentum and viscosity are also contained in the examples considered. For long wavelengths some general results are given.

INTRODUCTION

It has been shown in Jones & Prosperetti (1985), hereafter referred to as Part I, that a number of results concerning a class of two-phase flow models could be obtained by an analysis of the linear stability properties of their steady, uniform solutions. This approach is useful both in the initial assessment of specific models and as a guide to their further development. In Part I models including only first-order time and space derivatives were considered. In this paper that analysis is extended to deal with a broader class of models containing time and space derivatives of arbitrary order.

A number of such models can be found in the literature, such as those proposed by Arai (1980), Banerjee (1980), Sha & Slattery (1980), Smith (1980), Ramshaw & Trapp (1978), Ramson & Hicks (1984) and others. The generalization considered here is a natural extension of the class of models previously studied which offers the prospect of correcting its least acceptable feature, namely the independence of stability criteria from the wavelength of the perturbations. It is found that higher order derivatives lead to a wavenumber dependence, as expected, and make it possible to have, for instance, short-wavelength stability combined with long-wavelength instability. Although an averaged-equation model is not expected to provide more than a rough description of short-wavelength phenomena we consider stability on this scale to be an essential requirement for a realistic model. Numerically, this aspect has played a less important role due to the use of finite-size grids which limit the minimum resolvable wavelength.

The results of the general theory are applied to several models available in the literature for the description of stratified flow. The models contain a variety of physical effects such as surface tension, gravity, transverse momentum and viscosity. Their linear stability properties at different wavelengths are described in detail and stability boundaries involving the model parameters are obtained.

In the following developments the same approximations used and discussed in Part I are again adopted. Specifically, the phases are assumed to be individually incompressible, the flow is one-dimensional, and the energy equations are taken to be decoupled from the mass and momentum conservation equations and are therefore not considered. In particular this implies that no mass is exchanged between the phases.

†Present address: Department of Mechanical Engineering, The Johns Hopkins University, Baltimore, MD 21218, U.S.A.

THE CLASS OF MODELS CONSIDERED

The class of one-dimensional adiabatic two-phase models considered here is a broad extension of that of Part I.† The mass conservation equations are

$$\frac{\partial}{\partial t}(\epsilon_G \rho_G) + \frac{\partial}{\partial x}(\epsilon_G \rho_G V_G) = 0 \qquad [1a]$$

and

$$\frac{\partial}{\partial t}(\epsilon_L \rho_L) + \frac{\partial}{\partial x}(\epsilon_L \rho_L V_L) = 0, \qquad [1b]$$

where ρ, V and ϵ denote the density, velocity and volume fraction, respectively. Although we use subscripts G and L to distinguish the two phases, this does not imply that our results are limited to gas–liquid systems. The volume fractions satisfy

$$\epsilon_G + \epsilon_L = 1. \qquad [2]$$

As in Part I we now make the assumption of constant density for both phases to remove $\rho_{G,L}$ from [1a, b] which, combined, lead to $\partial U/\partial x = 0$ where the volume velocity U is defined by

$$U(t) = \epsilon_G V_G + \epsilon_L V_L. \qquad [3]$$

From [2] and [3], we find

$$\epsilon_G = \frac{U - V_L}{V_G - V_L} \quad \text{and} \quad \epsilon_L = \frac{U - V_G}{V_L - V_G}. \qquad [4]$$

We shall restrict our analysis to two-pressure models such that the phase pressures can be put in the form

$$p_{L,G} = p + P_{L,G}(\epsilon_{L,G}, V_{L,G}; \rho_{L,G}),$$

where p is some average pressure. Note that the $P_{L,G}$ can also depend on the derivatives of their arguments. Some examples of such models are considered later. With this assumption, for incompressible phases, the momentum equations can be written in the general form

$$\frac{\partial}{\partial t}(\epsilon_j V_j) + \frac{\partial}{\partial x}(\epsilon_j V_j^2) + \frac{\epsilon_j}{\rho_j}\frac{\partial p}{\partial x} = \epsilon_j(A_j + R_j), \quad j = \text{G,L}. \qquad [5]$$

Here A_j designates albegraic terms in the flow variables ϵ_j and V_j describing the effects of body force and steady drag (for an example see [4] of Part I). The symbol R_j represents a number of terms involving space and time derivatives of arbitrary order of ϵ_j and V_j such as are introduced by the modelling of differences in the phase pressures and of the effects of surface tension, viscosity, added mass, correlation terms, and others. A sufficiently general form of R_j for our purposes is

$$R_G = \sum_{j=\text{L,G}} \left[q_{Gj}\frac{\partial V_j}{\partial t} + r_{Gj}\frac{\partial V_j}{\partial x} + \sum_{m+n\geq 2}^{M,N} y_{Gj}^{(m,n)} \frac{\partial^{m+n} V_j}{\partial t^m \partial x^n} \right] + \begin{pmatrix}\text{terms non-linear in}\\ \text{the derivatives}\end{pmatrix} + \begin{pmatrix}\text{terms involving}\\ \dot{U}, \ddot{U}, \ldots\end{pmatrix}, \qquad [6]$$

with an analogous expression for R_L. Derivatives of the volume fractions appearing in $R_{G,L}$ have been eliminated by the use of [4]. The terms non-linear in the derivatives need not be shown explicitly since they vanish in the *linear* stability analysis of steady uniform flow which is the object of this paper. The quantities q_{Gj}, r_{Gj} and $y_{Gj}^{(m,n)}$ are functions of $V_{G,L}$ and $\epsilon_{G,L}$, specific forms of which are presented in the examples below. Note that for $y_{G,Lj}^{(m,n)} = 0$ the class of models considered in Part I is recovered.

†All symbols are the same as used in Part I, except for the substitution of ϵ for α to conform with the current journal nomenclature.

The same procedure as used in Part I allows us to combine the momentum equations [5] to find, with the aid of [3] and [4], an expression for the pressure gradient:

$$\frac{\partial p}{\partial x} = \tilde{\rho}\left[\epsilon_G(A_G + R_G) + \epsilon_L(A_L + R_L) + (V_L - U)\frac{\partial V_G}{\partial x} + (V_G - U)\frac{\partial V_L}{\partial x}\right], \tag{7}$$

where

$$\tilde{\rho} = \left(\frac{\epsilon_G}{\rho_G} + \frac{\epsilon_L}{\rho_L}\right)^{-1}. \tag{8}$$

The pressure gradient in [5] can be eliminated using this expression to find an alternative form for the momentum equations:

$$\frac{\partial V_G}{\partial t} + W\frac{\partial V_G}{\partial x} + \frac{\epsilon_L}{\epsilon_G}(V_G - W)\frac{\partial V_L}{\partial x} = \frac{\tilde{\rho}}{\rho_G}\dot{U} + \epsilon_L L \tag{9a}$$

and

$$\frac{\partial V_L}{\partial t} + \frac{\epsilon_G}{\epsilon_L}(V_L - W)\frac{\partial V_G}{\partial x} + W\frac{\partial V_L}{\partial x} = \frac{\tilde{\rho}}{\rho_L}\dot{U} - \epsilon_G L, \tag{9b}$$

in which

$$W = \left(\frac{\epsilon_L}{\rho_L}V_G + \frac{\epsilon_G}{\rho_G}V_L\right)\tilde{\rho} \tag{10}$$

and

$$L = \tilde{\rho}\left[\frac{1}{\rho_L}(A_G + R_G) - \frac{1}{\rho_G}(A_L + R_L)\right]. \tag{11}$$

Since the algebraic contributions to the momentum equations [5] are represented by $A_{G,L}$, without loss of generality, we can assume $R_{G,L} = 0$ when all the derivatives vanish. Thus in steady uniform flow [9a, b] reduce to $L = 0$ or

$$\rho_G A_G = \rho_L A_L. \tag{12}$$

It is obvious from [5] that this is just $\partial p/\partial x$.

LINEAR STABILITY ANALYSIS OF STEADY UNIFORM FLOW

Following Part I, we now investigate on the basis of [9a, b] the linear stability of steady uniform flow with velocities $\bar{V}_{G,L}$. We set

$$V_{G,L} = \bar{V}_{G,L} + \tilde{v}_{G,L}(x, t) \tag{13a}$$

and

$$U = \bar{U} + u(t), \tag{13b}$$

where $\tilde{v}_{G,L}$ and u are the small perturbations to the flow. To first order in these quantities, [9a, b] become

$$\frac{\partial \tilde{v}_G}{\partial t} + W\frac{\partial \tilde{v}_G}{\partial x} + \frac{\epsilon_L}{\epsilon_G}(V_G - W)\frac{\partial \tilde{v}_L}{\partial x} = \frac{\tilde{\rho}}{\rho_G}\dot{u} + \epsilon_L(L_G\tilde{v}_G + L_L\tilde{v}_L + L_u\dot{u}) \tag{14a}$$

and

$$\frac{\partial \tilde{v}_L}{\partial t} + \frac{\epsilon_G}{\epsilon_L}(V_L - W)\frac{\partial \tilde{v}_G}{\partial x} + W\frac{\partial \tilde{v}_L}{\partial x} = \frac{\tilde{\rho}}{\rho_L}\dot{u} - \epsilon_G(L_G\tilde{v}_G + L_L\tilde{v}_L + L_u\dot{u}). \tag{14b}$$

Here all the coefficients are evaluated for the unperturbed conditions but the overbars have been dropped. Equation [12] has been used to simplify the r.h.s.'s and $L_{G,L}$ and L_u are linear differential operators arising upon the insertion of [13a, b] into [1a, b] and [6] and linearization. The explicit form of $L_{G,L}$ is found to be

$$L_L = \gamma_L + \eta_L \frac{\partial}{\partial t} + \theta_L \frac{\partial}{\partial x} + \sum_{m+n \geq 2} Y_L^{(m,n)} \frac{\partial^{m+n}}{\partial t^m \partial x^n}, \quad [15a]$$

$$L_G = \gamma_G - \eta_G \frac{\partial}{\partial t} - \theta_G \frac{\partial}{\partial x} - \sum_{m+n \geq 2} Y_G^{(m,n)} \frac{\partial^{m+n}}{\partial t^m \partial x^n}, \quad [15b]$$

where

$$\gamma_G = \tilde{\rho} \frac{\partial}{\partial V_G} \left(\frac{A_G}{\rho_L} - \frac{A_L}{\rho_G} \right), \quad [16]$$

$$\eta_G = \tilde{\rho} \left(\frac{q_{LG}}{\rho_G} - \frac{q_{GG}}{\rho_L} \right) \quad [17]$$

and

$$\theta_G = \tilde{\rho} \left(\frac{r_{LG}}{\rho_G} - \frac{r_{GG}}{\rho_L} \right), \quad [18]$$

as in Part I and, furthermore,

$$Y_G^{(m,n)} = \tilde{\rho} \left(\frac{y_{LG}^{(m,n)}}{\rho_G} - \frac{y_{GG}^{(m,n)}}{\rho_L} \right) \quad [19]$$

with γ_L, η_L, θ_L, and $Y_L^{(m,n)}$ obtained by interchanging the indices L and G. For L_u we note that it is a differential operator on $\dot{u}$ with *constant coefficients* due to the assumed steadiness and spatial uniformity of the unperturbed flow.

Therefore, if one sets

$$\tilde{v}_{L,G} = v_{L,G}(x, t) + v'_{L,G}(t), \quad [20]$$

where $v'_{L,G}$ are defined as the solutions of

$$\frac{dv_G}{dt} = \left(\frac{\tilde{\rho}}{\rho_G} + \epsilon_L L_u \right) \dot{u} \quad [21a]$$

and

$$\frac{dv_L}{dt} = \left(\frac{\tilde{\rho}}{\rho_L} - \epsilon_G L_u \right) \dot{u}, \quad [21b]$$

then $v_{G,L}$ satisfy equations identical to [14a, b] but without the terms involving $\dot{u}$. An explicit example of this procedure can be found in Part I.

Note that the coefficients of the differential system [21a, b] are independent of x due to the assumed steadiness and uniformity of the base flow. The quantities $v'_{G,L}$ describe the response of the flow to perturbations in the total volume flux U which, being only a function of time, in practice is controlled by the inlet conditions. With this observation we can explicitly dispose of the boundary conditions to concentrate on [14a, b] with $\dot{u} = 0$.

In the standard fashion we set

$$v_{G,L} = U_{G,L} \exp[i(kx - \omega t)], \quad [22]$$

where k is the wavenumber of the perturbation and ω its angular frequency.

We note that since by the use of the transformation [20] the effect of inlet perturbations has been eliminated, the only meaningful sense in which the stability should be examined is as an initial-value problem. This implies real k and stability requires

$$\mathcal{I}m(\omega) \leq 0. \quad [23]$$

In Part I the quantity $c = i\omega/k$ was used in place of ω.

Upon substitution of [22] into [14a, b] (with $\dot{u} = 0$), one finds a linear homogeneous system in $U_{G,L}$, the solvability condition of which leads to the dispersion relation

$$(1+\epsilon_G\eta_L+\epsilon_L\eta_G)\omega^2-[i(\epsilon_G\gamma_L+\epsilon_L\gamma_G)+2kW+k(\epsilon_L\eta_G V_G+\epsilon_G\eta_L V_L)+k(\epsilon_G\theta_L+\epsilon_L\theta_G)]\omega$$
$$+ik(\epsilon_G\gamma_L V_L+\epsilon_L\gamma_G V_G)+k^2\left(\frac{\epsilon_G}{\rho_G}V_L^2+\frac{\epsilon_L}{\rho_L}V_G^2\right)+k^2(\epsilon_G\theta_L V_L+\epsilon_L\theta_G V_G)$$
$$+i\sum_{m+n\geq 2}^{M,N}\sum_{j=G,L}(1-\epsilon_j)(\omega-kV_j)Y_j^{(m,n)}(-i\omega)^m(ik)^n=0. \qquad [24]$$

Note that by [12] the factor $\tilde{\rho}$ need not be differentiated in this linear analysis. Furthermore, $\epsilon_L\gamma_G=\beta_{GG}$ and $\epsilon_G\gamma_L=\beta_{LL}$ in the notation of [27] of Part I. In the general case, [24] is a high-order polynomial in ω with complex coefficients and it is impossible to translate the stability condition [23] into explicit conditions involving the algebraic drag functions γ_k and the further terms of the model q_{jl}, r_{jl} and $y_{jl}^{(m,n)}$, as was done for the relatively simple case of Part I. Some information can, however, be derived from [24] in cases more general than those of Part I by examining the behaviour of $\omega(k)$ in the long-wavelength limit $k\to 0$. This will be done in the next section. The subsequent sections deal with more specific situations for which both short- and long-wavelength results can be obtained.

In the special case in which only first-order time derivatives appear in $R_{G,L}$, [24] is quadratic in ω and explicit stability conditions can be written down (see Part I). If, on the other hand, $R_{G,L}$ do not contain time derivatives but only *even-order* space derivatives a simple redefinition of $\eta_{G,L}$ and $\gamma_{G,L}$ shows that instability prevails unless $V_G=V_L$. An example is the incompressible model of Arai (1980) including viscosity but without added mass.

STABILITY IN THE LONG-WAVELENGTH LIMIT

The dispersion equation [24] is a polynomial of order M, say, and for stability each of its M roots must have a non-negative imaginary part. In the long-wavelength limit the behaviour of these roots must be examined as $k\to 0$.

A *necessary* stability condition can be derived by observing that [24] always has at least one root which tends to zero as $k\to 0$. By direct substitution it can be verified that such a root is

$$\omega_1=kV_\gamma-ik^2X+O(k^3), \qquad [25]$$

where V_γ is an average of the phase velocities weighted with the quantities $\epsilon_{G,L}$ and $\gamma_{L,G}$, related to the drag functions, and is defined by

$$V_\gamma=\frac{\epsilon_G\gamma_L V_L+\epsilon_L\gamma_G V_G}{\epsilon_G\gamma_L+\epsilon_L\gamma_G}. \qquad [26]$$

The quantity X in [25] is dependent only on the drag functions and on the coefficients of the lowest-order (first) space and time derivatives in the momentum equation [5]. The definition is

$$X=\Bigg[(1+\epsilon_G\eta_L+\epsilon_L\eta_G)V_\gamma^2-(2W+\epsilon_G\eta_L V_L+\epsilon_L\eta_G V_G+\epsilon_G\theta_L+\epsilon_L\theta_G)V_\gamma$$
$$+\tilde{\rho}\left(\frac{\epsilon_G}{\rho_G}V_L^2+\frac{\epsilon_L}{\rho_L}V_G^2\right)+\epsilon_G\theta_L V_L+\epsilon_L\theta_G V_G\Bigg]\cdot(\epsilon_G\gamma_L+\epsilon_L\gamma_G)^{-1}. \qquad [27]$$

The stability condition [23] for this root is just $X\geq 0$. Note that this necessary condition is sensitive to the form of the algebraic part $A_{G,L}$ of the drag functions which determine the quantities $\gamma_{G,L}$. Of course, further stability conditions will arise from the other roots of the dispersion relation making $X\geq 0$ only necessary.

It might be hoped that the instability of a non-hyperbolic first-order model could be corrected by the introduction of higher-order derivatives. However, as was shown in Part I, such a non-hyperbolic model would not satisfy $X\geq 0$ and so would be unstable in the long-wavelength limit. We note also that the condition $X \geqslant 0$, together with

$$\Gamma \equiv \frac{\epsilon_G \gamma_L + \epsilon_L \gamma_G}{1 + \epsilon_G \eta_L + \epsilon_L \eta_G} \leqslant 0, \qquad [28]$$

are equivalent to the two stability conditions of Part I, [44] and [49], which in that case were valid for all k.

The same conditions may be shown to arise in the long-wavelength limit under the restriction that the model under examination contains no time derivatives of second or higher order, i.e. $Y_j^{(m,n)} = 0$ for $m \geqslant 2$. The proof actually holds for the slightly more general case in which $\epsilon_G Y_L^{(m,n)} + \epsilon_L Y_G^{(m,n)} = 0$ for $m \geqslant 2$. Referring to [24], it is seen that this is now a quadratic equation in ω. One solution has the form [25], while the other is

$$\omega_2 = i\Gamma + O(k) \qquad [29]$$

with Γ defined by [28]. The stability condition for $k \to 0$ for this root is then [28]. Model features contributing terms of the type just discussed include the Arai (1980) viscosity term, Ramshaw & Trapp's (1978) surface tension term, the "hydrostatic" term of Rousseau & Ferch (1979) and Ardron (1980), and the 5E2P model of Ransom & Hicks (1984).

HORIZONTAL STRATIFIED FLOW: GRAVITY AND SURFACE TENSION EFFECTS

Horizontal stratified flow in a channel having a rectangular cross-section has been used by a number of investigators as a model problem for the investigation of several features of average two-phase flow equations. Ramshaw & Trapp (1978) have considered a model which includes surface tension effects and which, in the incompressible case, consists of the continuity equations [1a, b] and of momentum equations of the form

$$\frac{\partial V_j}{\partial t} + V_j \frac{\partial V_j}{\partial x} + \frac{1}{\rho_j}\frac{\partial p_j}{\partial x} = 0, \quad j = \mathrm{G, L}, \qquad [30]$$

with

$$p_L - p_G = -\sigma H \frac{\partial^2 \epsilon_L}{\partial x^2}. \qquad [31]$$

Here H is the height of the channel and σ the interfacial tension coefficient. To bring [30] into our general form [5] it is sufficient to define

$$p = \tfrac{1}{2}(p_G + p_L) \qquad [32]$$

to find

$$\frac{\partial V_j}{\partial t} + V_j \frac{\partial V_j}{\partial x} + \frac{1}{\rho_j}\frac{\partial p}{\partial x} = \frac{1}{2}\frac{\sigma}{\rho_j} H \frac{\partial^3 \epsilon_j}{\partial x^3}, \quad j = \mathrm{G, L}. \qquad [33]$$

Use of [4] then shows that

$$R_G = -\frac{1}{2\rho_G}\frac{\sigma H}{V_G - V_L}\left(\epsilon_G \frac{\partial^3 V_G}{\partial x^3} + \epsilon_L \frac{\partial^3 V_L}{\partial x^3}\right) \qquad [34]$$

up to terms non-linear in the derivatives. The form of R_L is obtained on interchanging the indices G and L.

The effect of gravity also gives rise to a pressure difference between the phases, as shown by Rousseau & Ferch (1979), Ardon (1980), Banerjee & Chan (1980) and others. In these models the pressure in the upper fluid, the gas, is given by

$$p_G = p_{GS} - \tfrac{1}{2} H \epsilon_G \rho_G g \qquad [35]$$

and in the lower fluid, the liquid, by

$$p_L = p_{LS} + \tfrac{1}{2} H \epsilon_L \rho_L g, \qquad [36]$$

where p_{GS} and p_{LS} are the values at the interface. This is just the hydrostatic approximation commonly used in hydraulics.

We can combine the effects of gravity and surface tension by assuming the p_{LS} and p_{GS} in these equations are related by [31]. If we then again let

$$p = \tfrac{1}{2}(p_{GS} + p_{LS}), \qquad [37]$$

we find momentum equations of the general form [5] with

$$R_G = -\frac{H}{V_G - V_L}\left[g\left(\epsilon_G \frac{\partial V_G}{\partial x} + \epsilon_L \frac{\partial V_L}{\partial x}\right) + \frac{1}{2}\frac{\sigma}{\rho_G}\left(\epsilon_G \frac{\partial^3 V_G}{\partial x^3} + \epsilon_L \frac{\partial^3 V_L}{\partial x^3}\right)\right]; \qquad [38]$$

a similar expression for R_L is obtained by an interchange of the indices. Again, only terms linear in the derivatives have been written down explicitly in [38]. Comparing with [6], we are then led to the identifications

$$r_{GL} = -Hg\frac{\epsilon_L}{V_G - V_L}, \quad r_{GG} = -Hg\frac{\epsilon_G}{V_G - V_L} \qquad [39]$$

and

$$y_{GL}^{(0,3)} = -\frac{\sigma}{2\rho_G}H\frac{\epsilon_L}{V_G - V_L}, \quad y_{GG}^{(0,3)} = -\frac{\sigma}{2\rho_G}H\frac{\epsilon_G}{V_G - V_L}; \qquad [40]$$

with the corresponding liquid quantities obtained by an interchange of the indices, all other coefficients being zero. From the definitions [16]–[19] we then find

$$\eta_G = \eta_L = 0, \qquad [41]$$

$$\theta_G = -\frac{H\tilde{\rho}}{\rho_G\rho_L}g(\rho_L - \rho_G)\frac{\epsilon_G}{V_G - V_L} = -\frac{\epsilon_G}{\epsilon_L}\theta_L \qquad [42]$$

and

$$Y_G^{(0,3)} = \frac{\sigma\tilde{\rho}}{\rho_G\rho_L}H\frac{\epsilon_G}{V_G - V_L} = -\frac{\epsilon_G}{\epsilon_L}Y_L^{(0,3)}. \qquad [43]$$

The dispersion relation [24] can now be written in the form

$$(\omega - kW)^2 - i(\epsilon_L\gamma_G + \epsilon_G\gamma_L)(\omega - kW) + ik\epsilon_G\epsilon_L(V_G - V_L)\tilde{\rho}\left(\frac{\gamma_G}{\rho_G} - \frac{\gamma_L}{\rho_L}\right) + \frac{k^2\epsilon_G\epsilon_L}{\rho_G\rho_L}\tilde{\rho}\left[\tilde{\rho}(V_G - V_L)^2 - Hg(\rho_L - \rho_G) - k^2\sigma H\right] = 0. \qquad [44]$$

If, as in the papers referred to above, the drag functions are disregarded, then $\gamma_G = \gamma_L = 0$, and the stability requirements reduce to just

$$\tilde{\rho}(V_G - V_L)^2 \leqslant [k^2\sigma + g(\rho_L - \rho_G)]H. \qquad [45]$$

This is the standard stability condition for the Kelvin–Helmholtz problem when the two fluids are of infinite vertical extent, and is the approximate form valid when $\epsilon_{G,L}Hk \gg 1$ (Lamb 1932; Ramshaw & Trapp 1978; Ardron 1980; Rousseau & Ferch 1979; Banerjee & Chan 1980).

The presence of drag modifies this condition to

$$\tilde{\rho}(V_G - V_L)^2 \leqslant [k^2\sigma + g(\rho_L - \rho_G)]H(1 - f^2), \qquad [46]$$

where f^2 is positive definite and is given by

$$f^2 = \tilde{\rho}\epsilon_G\epsilon_L\frac{\left(\dfrac{\gamma_G}{\rho_G} + \dfrac{\gamma_L}{\rho_L}\right)^2}{\dfrac{\epsilon_L}{\rho_G}\gamma_G^2 + \dfrac{\epsilon_G}{\rho_L}\gamma_L^2}. \qquad [47]$$

Hence, the addition of drag forces represented exclusively by algebraic terms, whatever their form, always has a destabilizing effect on the equations.

It is readily shown that $0 \leqslant f \leqslant 1$, the upper limit being attained for

$$\epsilon_G \gamma_L = \epsilon_L \gamma_G. \qquad [48]$$

When this relation holds the range of stability reduces to $V_G = V_L$. In the long-wavelength limit, $k \to 0$, [46] reduces to

$$|V_G - V_L| \leqslant \left[\frac{\rho_L - \rho_G}{\tilde{\rho}} Hg\,(1 - f^2)\right]^{\frac{1}{2}}, \qquad [49]$$

in agreement with [27], while the other condition [28] is

$$\epsilon_L \gamma_G + \epsilon_G \gamma_L \leqslant 0. \qquad [50]$$

In the opposite limit, $k \to \infty$, [46] ensures stability for any relative velocity (unless $f = 1$) since $\sigma > 0$.

HORIZONTAL STRATIFIED FLOW: ADDED MASS EFFECTS

In the preceding section a difference in the phase pressures was caused by surface tension and gravity. Banerjee (1980) has proposed a stratified flow model in which pressure differences caused by motion of the phases transverse to the direction of mean flow are approximately accounted for. In his formulation this effect, which is the added mass interaction appropriate for stratified flow, leads to the following momentum equations:

$$\frac{\partial V_G}{\partial t} + V_G \frac{\partial V_G}{\partial x} + \frac{1}{\rho_G}\frac{\partial p}{\partial x} - Hg\frac{\partial \epsilon_G}{\partial x} + \frac{1}{3}H^2 \epsilon_G \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial t} + V_G \frac{\partial}{\partial x}\right)\right]^2 \epsilon_G \Bigg] = 0 \qquad [51a]$$

and

$$\frac{\partial V_L}{\partial t} + V_L \frac{\partial V_L}{\partial x} + \frac{1}{\rho_L}\frac{\partial p}{\partial x} - Hg\frac{\partial \epsilon_G}{\partial x} + \frac{1}{3}H^2 \epsilon_L \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial t} + V_L \frac{\partial}{\partial x}\right)\right]^2 \epsilon_L \Bigg] = 0, \qquad [51b]$$

while the continuity equations are still [1a,b]. As in the previous section, H is the total height of the channel. By use of the definitions it is straightforward to show that

$$\left.\begin{aligned} &y_{GG}^{(2,1)} = \frac{1}{3}\frac{\epsilon_G^2 H^2}{V_G - V_L} = \frac{\epsilon_G}{\epsilon_L} y_{GL}^{(2,1)}, \quad y_{GG}^{(1,2)} = 2y_{GG}^{(2,1)} \cdot V_G, \\ &y_{GL}^{(1,2)} = 2y_{GL}^{(2,1)} \cdot V_G, \quad y_{GG}^{(0,3)} = y_{GG}^{(2,1)} \cdot V_G^2 \quad \text{and} \quad y_{GL}^{(0,3)} = y_{GL}^{(2,1)} \cdot V_G^2 \end{aligned}\right\} \qquad [52]$$

with $y_{LL}^{(m,n)}$ and $y_{LG}^{(m,n)}$ obtained by an interchange of G and L.

With these expressions and the $\theta_{G,L}$ given by [42] the dispersion relation [24] may be written as

$$\left[1 + k^2 Q\left(\frac{\epsilon_L}{\rho_G} + \frac{\epsilon_G}{\rho_L}\right)\right]\omega^2 - \left[i(\epsilon_G\gamma_L + \epsilon_L\gamma_G) + 2kW + 2k^3 Q\left(\frac{\epsilon_L}{\rho_G}V_L + \frac{\epsilon_G}{\rho_L}V_G\right)\right]\omega$$
$$+ ik(\epsilon_G\gamma_L V_L + \epsilon_L\gamma_G V_G) + k^2\tilde{\rho}\left(\frac{\epsilon_G}{\rho_G}V_L^2 + \frac{\epsilon_L}{\rho_L}V_G^2\right) - k^2\epsilon_G\epsilon_L\tilde{\rho}gH\left(\frac{1}{\rho_G} - \frac{1}{\rho_L}\right) + k^4 J = 0, \qquad [53]$$

where

$$Q = \tfrac{1}{3}\epsilon_G\epsilon_L H^2 \tilde{\rho} \qquad [54]$$

and

$$J = Q\left(\frac{\epsilon_L}{\rho_G}V_L^2 + \frac{\epsilon_G}{\rho_L}V_G^2\right). \qquad [55]$$

Since the model [51a,b] has been derived to describe waves of wavelength $> H$ it is expected to fail at short wavelengths. Indeed, it is easy to show that, as $k \to \infty$, [53] becomes

$$\left(\frac{\epsilon_G}{\rho_L} + \frac{\epsilon_L}{\rho_G}\right)\left(\frac{\omega}{k}\right)^2 - 2\left(\frac{\epsilon_L}{\rho_G}V_L + \frac{\epsilon_G}{\rho_L}V_G\right)\left(\frac{\omega}{k}\right) + \frac{\epsilon_L}{\rho_G}V_L^2 + \frac{\epsilon_G}{\rho_L}V_G^2 = 0 \qquad [56]$$

and the stability condition $\mathscr{I}m\,\omega \leqslant 0$ can only be satisfied if $(V_G - V_L)^2 \leqslant 0$, which is impossible in general. For numerical applications it may be useful to avoid this unsatisfactory behaviour of the equations at short wavelength, which can be achieved by the introduction of surface tension effects in the way shown in the previous section.

It is readily seen that the introduction of the term multiplied by σ in [33] into [51a,b] leads to the same dispersion equation [53] with J replaced by

$$J' = J - \frac{\sigma H \tilde{\rho} \epsilon_G}{\rho_L \rho_G}. \qquad [57]$$

The stability condition, as $k \to \infty$, is now

$$(V_G - V_L)^2 \leqslant \frac{3\sigma}{\epsilon_G \epsilon_L} H\left(\frac{\epsilon_G}{\rho_L} + \frac{\epsilon_L}{\rho_G}\right). \qquad [58]$$

For variable ϵ_G the r.h.s. has a minimum for $\epsilon_G = [1 + (\rho_G/\rho_L)^{1/2}]^{-1}$, where it has the value $3\sigma H(\rho_L^{-1/2} + \rho_G^{-1/2})^2$. The short-wavelength stability is thus substantially improved.

At long wavelengths one stability condition is, as always, $X \geqslant 0$ with X defined by [27]. The second condition is [28] since, as is readily verified, $\epsilon_G Y_L^{(m,n)} + \epsilon_L Y_G^{(m,n)} = 0$ for $m = 2$ in the present model so that the theory leading to [29] applies. Surface tension does not affect either condition, nor does the added mass term.

ANOTHER MODEL FOR HORIZONTAL STRATIFIED FLOW

A further model for stratified flow has recently been proposed by Ransom & Hicks (1984) on the basis of an analysis of transverse momentum effects, the physical content of which is quite different from that of Banerjee's model considered in the previous section. Although this model appears to be physically incorrect because of the critical role which it ascribes to compressibility, even at low speeds, its mathematical structure is interesting for our purposes. Our conclusion is that in the limit we consider, even aside from its physical content, the model must be rejected because it is unconditionally unstable. The version of Ransom & Hicks' model in which the momentum equations are independent of the energy equations (termed by them the 5E2P model) is closest to our analytical starting point [5] and will be analysed in terms of the results derived in the previous sections. In the 5E2P model the continuity equations are identical with [1a,b] up to terms of order $c_{G,L}^{-2}$, where c is the speed of sound. The momentum equations in our notation are

$$\frac{\partial}{\partial t}(\rho_G \epsilon_G V_G) + \frac{\partial}{\partial x}(\rho_G \epsilon_G V_G^2) + \epsilon_G \frac{\partial p}{\partial x} + (p_G - p)\frac{\partial \epsilon_G}{\partial x} = 0 \qquad [59a]$$

and

$$\frac{\partial}{\partial t}(\rho_L \epsilon_L V_L) + \frac{\partial}{\partial x}(\rho_L \epsilon_L V_L^2) + \epsilon_L \frac{\partial p}{\partial x} + (p_L - p)\frac{\partial \epsilon_L}{\partial x} = 0, \qquad [59b]$$

in which the interface pressure p is defined as

$$\left.\begin{aligned} p &= (p_G a_G^{-1} + p_L a_L^{-1})(a_G^{-1} + a_L^{-1})^{-1} \\ p &= \lambda p_G + (1 - \lambda) p_L, \end{aligned}\right\} \qquad [60]$$

where $0 \leqslant \lambda \leqslant 1$ is a constant and the relationship between the phase pressures p_G and p_L is provided by the PDE

$$\frac{\partial \epsilon_G}{\partial t} + \frac{V_G + V_L}{2} \cdot \frac{\partial \epsilon_G}{\partial x} = \mathrm{K}^{-1}(p_G - p_L), \qquad [61]$$

in which K is a second constant.

In the original Ransom & Hicks (1984) model $\lambda = a_G^{-1}/(a_G^{-1} + 2a_L^{-1})$, $\mathrm{K} = H(a_G + a_L)$, with $a_{G,L}$ representing the acoustic impedances of the pure phases. In Ransom & Hicks' model the phases are compressible, whereas our analysis assumes incompressibility. Formally, however, we can take the incompressible limit $a_{G,L} \to \infty$, provided $H \to 0$ at the same time in such a way that K remains constant. In this limit, which corresponds to the case in which the acoustic transit time across the channel width is small compared with the period of the surface disturbances, it is appropriate to apply our incompressible analysis.

In the case of constant phase densities [59a,b]–[61] may be combined to produce momentum equations of the form [5] in which

$$R_G = -\frac{(1-\lambda)}{\rho_G \epsilon_G}\frac{\partial}{\partial x}[\epsilon_G(p_G - p_L)] = -\frac{K(1-\lambda)}{\rho_G \epsilon_G}\frac{\partial}{\partial x}\left[\epsilon_G\left(\frac{\partial}{\partial t} + \frac{V_G + V_L}{2}\frac{\partial}{\partial x}\right)\epsilon_G\right] \qquad [62a]$$

and

$$R_L = -\frac{K\lambda}{\rho_L \epsilon_L}\frac{\partial}{\partial x}\left[\epsilon_L\left(\frac{\partial}{\partial t} + \frac{V_G + V_L}{2}\frac{\partial}{\partial x}\right)\epsilon_L\right]. \qquad [62b]$$

As already remarked, the terms non-linear in the derivatives vanish in the linear theory due to the assumed steady uniform character of the base flow and these expressions can, for the present purpose, be simplified to

$$R_G = -\frac{K(1-\lambda)}{\rho_G}\left(\frac{\partial^2}{\partial t \partial x} + \frac{V_G + V_L}{2}\frac{\partial^2}{\partial x^2}\right)\epsilon_G \qquad [63a]$$

and

$$R_L = -\frac{K\lambda}{\rho_L} H\left(\frac{\partial^2}{\partial t \partial x} + \frac{V_G + V_L}{2}\frac{\partial^2}{\partial x^2}\right)\epsilon_L. \qquad [63b]$$

Proceeding as before, the dispersion relation [24] for the 5E2P model is found to be

$$\omega^2 - [i(\epsilon_G \gamma_L + \epsilon_L \gamma_G) + 2kW]\omega + ik(\epsilon_G \gamma_L V_L + \epsilon_L \gamma_G V_G) + k^2 \tilde{\rho}\left(\frac{\epsilon_L}{\rho_L} V_G^2 + \frac{\epsilon_G}{\rho_G} V_L^2\right)$$
$$+ i\omega k^2 \frac{\tilde{\rho} K \epsilon_G \epsilon_L}{\rho_G \rho_L} - ik^3 \tilde{\rho}\frac{K \epsilon_G \epsilon_L}{\rho_G \rho_L}\cdot\frac{V_G + V_L}{2} = 0. \qquad [64]$$

With the substitutions

$$\gamma_L' = \gamma_L - k^2 \frac{\tilde{\rho} K \epsilon_L}{2\rho_G \rho_L} \quad \text{and} \quad \gamma_G' = \gamma_G - k^2 \frac{\tilde{\rho} K \epsilon_G}{2\rho_G \rho_L}, \qquad [65]$$

[64] may be written in the form

$$\omega^2 - [i(\epsilon_G \gamma_L' + \epsilon_L \gamma_G') + 2kW]\omega + ik(\epsilon_G \gamma_L' V_L + \epsilon_L \gamma_G' V_G) + k^2 \tilde{\rho}\left(\frac{\epsilon_L}{\rho_L} V_G^2 + \frac{\epsilon_G}{\rho_L} V_L^2\right) = 0, \qquad [66]$$

which has the same form as the stability equation of Part I and readily leads to the stability conditions

$$\epsilon_G \gamma_L + \epsilon_L \gamma_G \leqslant k^2 \frac{\tilde{\rho} K \epsilon_G \epsilon_L}{2\rho_G \rho_L} \qquad [67a]$$

and

$$(V_G - V_L)^2 \tilde{\rho} \epsilon_G \epsilon_L \left(\frac{\epsilon_L}{\rho_G}\gamma_G'^2 + \frac{\epsilon_G}{\rho_L}\gamma_L'^2\right) \leqslant 0. \qquad [67b]$$

Clearly [67b] cannot be satisfied for any k. Hence Ransom & Hicks' 5E2P model predicts that no uniform steady flows are possible. This seems unphysical and is an interesting result of our analysis since Ransom & Hicks (1984) were able to prove that their model has real characteristics. This is a confirmatory instance of the well-known fact that reality of characteristics is necessary but not in general sufficient for stability. We conclude that this model is not only physically incorrect but also has unacceptable stability properties.

CONCLUSIONS

In this paper we have considered a general class of one-dimensional, two-phase flow models. This class of models is sufficiently broad to accommodate a variety of physical phenomena such as surface tension, viscosity, correlation effects and added mass. Furthermore, unequal phase pressures can be included to the extent that their difference can be expressed in terms of the other flow variables and their time and space derivatives. Extending the methods of Part I we have examined steady uniform flows and derived general linear stability criteria for them. The introduction of derivatives of orders higher than the first causes the stability criteria to depend on the wavelength of the perturbation. Nevertheless one of the criteria of Part I is a necessary condition for stability in the long-wavelength limit. As a consequence, we conclude that the long-wavelength stability behaviour of a first-order hyperbolic model cannot be improved by the addition of higher-order derivatives.

In our formulation drag effects between the phases and with structures enter in a natural way and prove to be decisive for long-wavelength stability in many instances, as shown in the cases considered.

Our approach leads to a relatively straightforward technique for the evaluation of the stability features of specific models. The utility of this technique has been illustrated by its application to several current models for stratified flow.

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